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# A method for the approximate solution of quasi-static problems for hardening bodies☆

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**Abstract**

A method for the approximate solution of quasi-static problems for hardening elastoplastic bodies is proposed. The constitutive relation of the model is taken in the form of a variational inequality. An approximate solution of the initial problem is constructed in time steps and, by means of the finite element method, is reduced to the solution of a system of two variational inequalities in corresponding finite-dimensional space. It is shown that the solution of this system is equivalent to finding the saddle point of the corresponding quadratic functional. To find the saddle point, Udzawa's algorithm is used, by means of which the process of finding the velocity vector and stress tensor reduces to the successive calculation of these quantities: the velocity vector is determined from the variational inequality corresponding to the equilibrium equation, and the stress tensor is determined from the variational inequality corresponding to the constitutive relation. The latter inequality is reduced to a certain non-linear equation containing the operation of projection onto a closed convex set corresponding to the elastic strains of the medium. In turn, the solution of the non-linear equation is constructed using the method of successive approximations. To illustrate the use of the proposed method, the one-dimensional problem of the quasi-static deformation of a cylindrical tube under a load applied to its internal surface is considered.


The model of a hardening elastoplastic body (HEPB), considered in the mechanics of deformable rigid bodies, differs substantially from other models in that the relations between the strain rate tensor and the stress tensor and its derivatives (the constitutive relations) are written in the form of an alternative system of equations and inequalities. By means of the inequalities, the region in stress space where the behaviour of the rigid body is described by the laws of the theory of elasticity is specified, and also the conditions for loading and unloading when the body is in the plastic state.

Formally, the system of constitutive relations can be reduced to a certain system of equations with discontinuous coefficients. Hence, the well-known method of the linearization of non-linear equations that is normally used to obtain a finite-dimensional approximation of the constitutive relations cannot be used. In many studies dealing with this problem, the stress correction procedure proposed by Wilkins[1,2] is used.

We will explain the basics of this procedure using the example of an ideally plastic body with the von Mises flow condition, when the flow surface in the space of principal stresses is a cylinder. The calculation of the stresses at each

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point of the body at the next time step can be presented in the form of the following algorithm. To begin with, the stress increments $\Delta\sigma$ are calculated from the strain increments using Hooke's law. Then, the stresses at the next step are calculated as the sum of $\Delta\sigma$ and the stress values at a given instant of time. If the stresses calculated in this way, $\sigma^*$, exceed the yield point, i.e. lie outside the cylinder, the stresses must be corrected. To this end, a perpendicular is dropped from the point $\sigma^*$ to the flow surface, and their point of intersection is found. The stresses corresponding to this point are taken as the stresses corresponding to the next time step. Grigoryan, in a comment on,[1] showed that it is possible to obtain the same stress values at the next time step by means of a finite difference approximation of the Prandtl–Reuss equations. Thus, the stress correction procedure has been rigorously substantiated for ideally plastic bodies with the von Mises flow condition.

To our knowledge, an analysis of this kind has not been carried out for HEPBs. However, owing to its simple geometrical interpretation, the stress correction procedure has begun to be used fairly often to solve many practical problems concerning HEPBs.

From the physical viewpoint this method has been fully substantiated. At the same time, the question naturally arises as to whether it is impossible to find another method in which the operation of projection onto a closed set in six-dimensional stress space, which is essentially a correction procedure, would follow naturally from the mathematical formulation of the problem and be rigorously substantiated.

The present paper will address one of these approaches.[3] It is based on the variational formulation of evolutionary problems, modelling quasi-static processes of the deformation of HEPBs, when describing which, as is well known, it is possible to ignore inertia terms in the equations of motion. Below, for brevity, such evolutionary problems will also be referred to as "quasi-static". The substance of this method can be briefly described as follows. The initial problem is written in the form of two variational inequalities: one in stress space, the other in velocity space. The time interval in which the solution is constructed is divided into a finite number of intervals. For each instant of time we approximate the velocity vector and stress tensor components using the corresponding finite-element spaces. The time derivatives are approximated by finite-difference relations, and the two inequalities in finite-dimensional space are associated with the initial system of variational inequalities.

The solution of this system of variational inequalities is constructed using Udzawa's iteration algorithm for finding the saddle points of non-linear functionals. The main advantage of this algorithm is that the determination of the two unknown quantities reduces to the alternate determination first of one quantity and then of the other. It can be shown (see the Appendix) that this iteration process reduces to solving a finite-dimensional problem. Moreover, the solution of the finite-dimensional problem reduces to solving the initial variational problem. In order to determine the stress tensor at each iteration, two methods can be used. One method for determining the stresses is based on the fast the variational inequality is equivalent to the problem of minimizing the quadratic function on a closed convex set. The other method, proposed in the present paper, leads to solving the non-linear equation by the method of successive approximations, and here, in determining each iteration, the operation of projection onto the loading surface is used.

The paper by Johnson[3] is one of the last in a series of papers by him that have been devoted to proving existence theorems and to constructing numerical methods for solving quasi-static problems of the theory of ideally plastic bodies and HEPBs. It differs from other studies of this series in being unnecessarily concise, which makes it difficult to understand a number of the assertions made. Furthermore, it contains a number of misprints that make individual mathematical calculations in the proofs of the theorems and lemmas unintelligible. One of the purposes of the present paper is to give, where possible, a complete description and to correct the misprints in the part concerning the construction of the approximate method and the proof of its convergence.

## 1. The mathematical model of hardening elastoplastic bodies

We will first introduce some notation. We will regard the space of symmetrical second-rank tensors as a six-dimensional Euclidean space with a scalar product

$$(\sigma, \tau) \equiv \sigma : \tau = \sigma_{ij}\tau_{ji}, \quad i, j = 1, 2, 3$$

The linear operators in this space are written using fourth-rank tensors $A$ according to the formula

$$\varepsilon = A\sigma \Leftrightarrow \varepsilon_{ij} = A_{ijkl}\sigma_{kl}, \quad i, j, k, l = 1, 2, 3$$

Below, we will examine the fourth-rank tensor $A = \{A_{ijkl}\}$ characterizing only the elastic behaviour of the medium, i.e. that for which the strains are reversible. It possesses the property of symmetry and positive definiteness

$$A_{ijkl} = A_{jikl} = A_{klij}; \quad (A\sigma : \sigma) \geq \alpha(\sigma : \sigma), \quad \alpha > 0$$

For an isotropic body, the components of the tensor $A$ can be represented in the form

$$A_{ijkl} = -\frac{\nu}{E}\delta_{ij}\delta_{kl} + \frac{1+\nu}{2E}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

where $E$ is Young's modulus, $\nu$ is Poisson's ratio and $\delta_{ij}$ is the Kronecker delta. We will denote by $v = v(x)$ and $\sigma = \sigma(x)$ the velocity and stress fields respectively. The components of the strain rate tensor $e(v)$ have the form

$$e_{ij}(v) = (\partial v_i/\partial x_j + \partial v_j/\partial x_i)/2$$

We will now formulate a mathematical model describing the behaviour of the constitutive relations of HEPBs. We will proceed on the basis that a certain initial state (initial configuration) of the rigid body has been specified in which the internal stresses vanish. Below we will assume that, in six-dimensional stress space, a certain family of regions of elastic behaviour of the rigid body has been specified: whatever the stresses, within these regions the process of deformation is reversible, and the relation between the stress and strain tensors is given by Hooke's law. The boundary of the regions is known as the loading surface. In HEPB theory it is assumed that the family of loading surfaces is specified using certain quantities that are called hardening parameters. These parameters remain constant in reversible processes.

In the general case the equation of the loading surface is written in the form

$$F(\sigma, \chi, T, \mu) = 0$$

Here, $\chi = \{\chi_1, \chi_2, \ldots, \chi_m\}$ represents the hardening parameters, $T$ is the temperature and $\mu = \{\mu_1, \mu_2, \ldots, \mu_m\}$ is a certain number of parameters of a different physicochemical nature. When $F(\sigma, \chi, T, \mu) < 0$, the body behaves elastically. The selection of a particular quantity as the hardening parameters is based upon physical considerations.

The behaviour of many important practical deformable rigid bodies can be described using two particular models: a model of isotropic HEPBs and a model of translational HEPBs.

In the former case, the loading surface equation has the form

$$F(\sigma, \chi, T) = f(\sigma, T) - H(\chi, T) = 0$$

where $\chi$ is the hardening parameter, which should increase monotonically in irreversible processes; the function $H = H(\chi, T)$ increases monotonically as $\chi$ increases and is termed the loading limit. The "plastic work" (Taylor, Quinney, Schmidt)

$$\chi = \int\frac{(\sigma : e^p)}{f(\sigma, T)}dt$$

or Odquist's parameter

$$\chi = \int(e^p : e^p)^{1/2}dt$$

(which is a measure of the accumulation of plastic strains) can be taken as this parameter. Here, $e^p = e(v) - A\dot{\sigma}$ is the strain rate tensor of plastic deformation. The dot denotes a time derivative.

In the case of a translational HEPB, the loading surface equation is written in the form

$$F(\sigma, \chi, T) = f(\sigma - \chi, T) = 0$$

Here, the hardening parameter is a second-rank tensor.

In order to derive the constitutive relations of the HEPB model, we will follow the thermodynamic approach.[4–7] Using this approach, we will regard the components of the plastic strain tensor and the hardening parameter as internal parameters of state. Below we will limit ourselves to an examination of isothermal processes of deformation under small strains. We will assume that the free (specific) energy $F$ depends on the complete strain tensor $\varepsilon$, the plastic strain tensor $\varepsilon^p$, and the hardening parameter $\chi$ and is represented in the form

$$F = (A^{-1}(\varepsilon - \varepsilon^p) : (\varepsilon - \varepsilon^p)) + \gamma\chi^2/2, \quad \gamma > 0$$

Below we will regard the strain rate tensor of plastic deformation $e^p$ ($e^p \approx \dot{\varepsilon}^p$) and the rate of change, $\dot{\chi}$, of the hardening parameter $\chi$ as generalized thermodynamic flows, and we will introduce the notation $\Psi = (e^p, \dot{\chi})$. The following generalized thermodynamic forces correspond to these

$$\mathcal{R} = (\sigma, r){:}\ e^p \to \sigma, \quad \dot{\chi} \to r = \partial F/\partial\chi = \gamma\chi$$

by means of which the entropy production rate due to irreversible internal processes $W$ is represented in the form

$$W = \sigma : e^p + r\dot{\chi} = (\mathcal{R}, \Psi)$$

In order to establish the constitutive relations for HEPBs, we will proceed on the basis of the hypothesis of "dissipation normality".[8] According to this hypothesis, for any elastoplastic medium, it is possible to specify a dissipation function $\varphi = \varphi(e^p, \dot{\chi})$, by means of which the relation between the generalized thermodynamic forces $\mathcal{R}$ and flows $(e^p, \dot{\chi})$ is established in the form

$$\mathcal{R} \in \partial\varphi \tag{1.1}$$

where $\partial\varphi(e^p, \dot{\chi})$ is the subdifferential[7–9] of the function $\varphi = \varphi(e^p, \dot{\chi})$. In the case of a smooth function, the subdifferential is the gradient of the function, i.e. $\partial\varphi(\mathcal{R}) = (\partial\varphi/\partial\varepsilon^p, \partial\varphi/\partial\dot{\chi})$.

The function $\varphi = \varphi(e^p, \dot{\chi})$ should be convex and semi-continuous from below.[6,8] In this case the Clausius–Duhem inequality, which is equivalent to the second law of thermodynamics – the law of increase in entropy in irreversible processes,[4,5] will be satisfied. If the Legendre–Fenchel transformation $\varphi^* = \varphi^*(\mathcal{R})$ of the function $\varphi = \varphi(e^p, \dot{\chi}) = \varphi(\Psi)$ is introduced, according to the formula

$$\varphi^*(\mathcal{R}) = \sup_{\Psi}((\mathcal{R}, \Psi) - \varphi(\Psi)) = \sup_{t,s}(\sigma : t + rs - \varphi(t, s)), \quad \text{где} \quad t = e^p, \quad s = \dot{\chi}$$

then relation (1.1) can be inverted and written in the form[9]

$$(e^p, \dot{\chi}) \in \partial\varphi^*(\mathcal{R}) \tag{1.2}$$

The Legendre–Fenchel transformation is an extension of the Legendre transformation[10] to the case of infinite-dimensional normed spaces. In the case of finite-dimensional Euclidean spaces, it is not difficult to show that, for the functions being differentiated, it is identical with the Legendre transformation.

In the theory of HEPB, the indicator function

$$\vartheta_B(\mathcal{R}) = \begin{cases} 0, & \mathcal{R} \in B \\ \infty, & \mathcal{R} \notin B \end{cases}$$

of the closed complex set $B$, within which the behaviour of the body is described by the laws of elasticity, is selected as the function $\varphi^* = \varphi^*(\mathcal{R})$. In this case, relations (1.2) are equivalent to the inequality

$$(e^p : (\tau - \sigma)) - \gamma\dot{\chi}(\eta - \chi) \le 0, \quad \forall(\tau, \eta) \in B \tag{1.3}$$

It is this inequality that is adopted below as the constitutive relation.

Note that, in a number of books,[11–15] other principles are used that make it possible to obtain constitutive relations: the von Mises Maximum Dissipation Principle, Drucker's axiom, or the associated flow law,[16] where the above-mentioned Taylor–Quinney–Schmidt or Odquist parameters are adopted as the hardening parameter.

It can be shown that the proposed formulation (1.2) agrees with Drucker's axiom and gives a more general relation between the rate of change in the hardening parameter $\dot{\chi}$ and the parameters of state $\mathcal{R} = (\sigma, \chi)$.

## 2. Formulation of the mathematical problem

We will assume that hardening is characterized by a single hardening parameter $\chi$, and we will consider jointly the stress tensor and the hardening parameter as a point in seven-dimensional space $(\sigma, \chi) \in R^6 \times R$. We will introduce in this space the closed convex set by means of the relation

$$B := \{(\sigma, \chi) | F(\sigma, \chi) \le 0\}$$

At the points of the set $B$, where the inequality $F(\sigma, \chi) < 0$ is satisfied, the behaviour of the rigid body is described by the laws of elasticity. Accordingly, we will introduce the functional space $H$ formed by the pairs $(\sigma, \chi)$, $H: = \{(\sigma, \chi)\}$. The set of permissible pairs of the stress field $\sigma = \sigma(x)$ and the hardening parameter field $\chi = \chi(x)$ is spcified by means of the relation

$$P := \{(\sigma, \chi) \in H | (\sigma(x), \chi(x)) \in B, \forall x \in \Omega\}$$

We will now give a mathematical formulation of the problem. We will assume that, on a part $\Gamma_u$ of the boundary of region $\Omega$, zero displacements are specified, and on a part $\Gamma_\sigma$ the external forces $P_N$, $\Gamma_u \cup \Gamma_\sigma = \partial\Omega$ are given ($\partial\Omega$ is the boundary of region $\Omega$). Note that, by a standard method, it is possible also to consider the case of non-zero conditions on $\Gamma_u$.

In formulating the problem, we will start from the variational formulation of the constitutive relations. We will use $V$ to denote the set of permissible velocity fields, i.e. those fields that vanish on the surface $\Gamma_u$. The problem can be formulated as follows.

**Problem 1.** It is required to find the velocity field $\upsilon = \upsilon(x) \in V$, the stress field $\sigma(x)$, and the hardening parameter field $\chi(x)$, $(\sigma, \chi) \in P$, such that the relations

$$(\sigma : e(w)) - (P_N, w)_s = (g, w), \quad \forall w \in V$$

$$((e(\upsilon) - A\dot{\sigma}) : (\tau - \sigma)) - \dot{\chi}(\eta - \chi) \le 0, \quad \forall (\tau, \eta) \in P$$

$$\sigma(0, x) = \sigma_0(x)$$

are satisfied. Here, the first relation – the variational equality – follows from the equilibrium equations and is termed the principle of virtual work, $g$ is the density of the distributed forces, $(P_N, w)_s$ is the integral over the surface $\Gamma_\sigma$, $P_N$ is the force on the surface $\Gamma_\sigma$, and $\sigma_0(x)$ is a certain specified function.

Below, the main focus of attention will be upon the construction of a numerical solution of the problem formulated above by means of the finite element method. To this end, we will examine the corresponding problem in finite-dimensional space, approximating the functional spaces $V$ and $H$ by spaces of finite elements $V_h$ and $H_h$ and replacing the time derivative with a finite-difference relation.

We will construct the solution in the segment $[0, T]$; we will divide it into $N$ parts, using $\Delta$ to denote the time step, $\Delta = T/N$, $t_n = n\Delta$ ($n = 1, 2, \ldots, N$). Then, in the region $\Omega$, we will introduce the triangulation $T_h = \cup_\nu T_\nu^h$. Here, $T_\nu^h$ ($\nu = 1, \ldots, M$) is a triangle from the division of the region $\Omega$, and $M$ is the number of triangles. It is assumed that the triangulation satisfies the normal conditions of regularity.[17]

We will approximate the space $V$ using the space of piecewise-linear functions $V_h$, and the space $H$ using the space of piecewise-constant functions $H_h$. The set $P$ will be replaced by the set

$$P_h = P \cap H_h$$

Any element $\bar{\sigma}_h$ of the space $H_h$ can be represented in the form

$$\bar{\sigma}_h(x) = \sum_{\nu=1}^{k} \bar{\sigma}_\nu \theta_\nu(x), \quad \bar{\sigma}_\nu = (\sigma_\nu, \chi_\nu), \quad \theta_\nu(x) = \begin{cases} 1, & x \in T_\nu^h \\ 0, & x \notin T_\nu^h \end{cases}$$

where $k$ is the number of subdivision elements and $\theta_\nu(x)$ is the characteristic function of the element $T_\nu^h$. The scalar product of any two elements $\bar{\sigma}_h, \bar{\tau}_h \in H_h$ is represented in the form of the sum

$$(\bar{\sigma}_h, \bar{\tau}_h) = \sum_{\nu=1}^{k} (\bar{\sigma}_\nu, \bar{\tau}_\nu)\left|T_\nu^h\right|$$

Here, $\left|T_\nu^h\right|$ is the measure of the finite element $T_\nu^h$, $(\bar{\sigma}_\nu, \bar{\tau}_\nu) = (\sigma_\nu, \tau_\nu) + \xi_\nu\eta_\nu$ if $\bar{\sigma}_\nu = (\sigma_\nu, \xi_\nu)$, $\bar{\tau}_\nu = (\tau_\nu, \eta_\nu)$. We will also give the formula for the distance from any point $\bar{\sigma}_h = (\sigma, \chi) \in H_h$ to the set $BP_h$:

$$d_B(\bar{\sigma}_h) = \min_{(\tau_\nu, \eta_\nu)\in B} \sqrt{\sum_{\nu=1}^{k} \left(|\sigma_\nu - \tau_\nu|^2 + |\chi_\nu - \eta_\nu|^2\right)\left|T_\nu^h\right|}$$

## 3. Approximation of the exact solution

We will now formulate the following problem in the finite-dimensional space $V_h \times H_h$.

**Problem 2.** It is required to find sequences $v^n \in V_h$, $(\sigma^n, \chi^n) \in P_h$ $(n = 1, 2, \ldots, N)$ such that the relations

$$\begin{aligned}
&(e(v^n) - A(\sigma^n - \sigma^{n-1})/\Delta):(\tau - \sigma^n) - \gamma(\chi^n - \chi^{n-1})(\eta - \chi^n)/\Delta \le 0, \quad \forall(\tau, \eta) \in P_h \\
&(\sigma^n : e(w)) - (g, w) - (P_N, w)_s = 0, \quad \forall w \in V_h \\
&\sigma(0, x) = \sigma_0(x)
\end{aligned} \tag{3.1}$$

are satisfied.

Thus, if, when $t = t_{n-1}$, the quantities $(\sigma^{n-1}, \chi^{n-1})$ are known, then $(\sigma^n, \chi^n)$ and $v^n$ can be determined from the solution of variational Problem 2. In order to obtain an approximate solution of Problem 1 for any $t \in [0, T]$, we will interpolate linearly in each time segment $[t_{k-1}, t_k]$ $(k = 1, 2, \ldots, N)$, successive values of $v^{k-1}$, $v^k$, and also $\sigma^{k-1}$, $\sigma^k$ and $\chi^{k-1}$, $\chi^k$.

We will introduce the notation $\bar{\sigma}^n = (\sigma^n, \chi^n)$. Then, the unknown quantities $v^n$ and $\bar{\sigma}^n$ can be regarded as the coordinates in the direct product of spaces $V_h \times H_h$. It can be shown that Problem 2 is equivalent to finding the saddle point of the functional

$$L(\bar{\sigma}, w) = [(A(\sigma^n - \sigma^{n-1}):\sigma) + \gamma(\chi^n - \chi^{n-1})\chi]/(2\Delta) - (e(w):\sigma) + (g, w) + (P_N, w)_s$$

which is convex with respect to variable $\bar{\sigma}$ and concave with respect to $w$.

Remember that the point $(\bar{\sigma}_0, w_0)$ is called the saddle point of the functional $L(\bar{\sigma}, w)$ if the following conditions are satisfied

$$L(\bar{\sigma}, w_0) > L(\bar{\sigma}_0, w_0) > L(\bar{\sigma}_0, w)$$

For the functionals being differentiated, these conditions are equivalent to the variational inequalities

$$\left(\frac{\partial L(\bar{\sigma}_0, w_0)}{\partial \bar{\sigma}} : (\bar{\tau} - \bar{\sigma}_0)\right) \ge 0, \quad \forall \bar{\tau} \in P_h; \quad \left(\frac{\partial L(\bar{\sigma}_0, w_0)}{\partial w} : (w - w_0)\right) \le 0, \quad \forall w \in V_h \tag{3.2}$$

If all the transformations are carried out here and $\bar{\sigma}^n$ and $v^n$ are substituted for $\bar{\sigma}_0$ and $w_0$, we will obtain exactly the variational Problem 2. Taking this into account, we will change to constucting an iteration procedure that will enable us to find the saddle point of the functional $L = L(\bar{\sigma}, w)$ and thereby obtain the numerical solution of Problem 2.

## 4. Algorithm for the approximate solution

We will seek the saddle point of the functional $L = L(\bar{\sigma}, w)$ using Udzawa's algorithm. It consists of constructing two convergent series $\{\bar{\sigma}_j^n\}$ and $\{v_j^n\}$ $(j=0, 1, 2, \ldots)$ that satisfy the following conditions

$$[(A(\sigma_j^n - \sigma^{n-1}):(\tau - \sigma_j^n) + \gamma(\chi_j^n - \chi^{n-1})(\eta - \chi_j^n))]/\Delta - (e(v_{j-1}^n):(\tau - \sigma_j^n)) \geq 0, \ \forall(\tau, \eta) \in P_h \tag{4.1}$$

$$(e(v_j^n):e(w)) = (e(v_{j-1}^n):e(w)) + \rho[(g^n, w) - (e(w):\sigma_j^n) + (P_N, w)_s], \quad \forall w \in V_h \tag{4.2}$$

where $\rho > 0$ is a parameter that will be evaluated in the Appendix below.

This algorithm can be represented schematically in the form of the construction of the following sequence ($v_0^n$ is selected arbitrarily from the set of permissible values)

$$v_0^n \to \bar{\sigma}_1^n \to v_1^n \to \bar{\sigma}_2^n \to \ldots \to v_j^n \to \bar{\sigma}_{j+1}^n \to v_{j+1}^n \to \ldots$$

Thus, the terms of the series $\bar{\sigma}_j^n$ are determined from the variational inequality (4.1) for known $v_{j-1}^n$, and the terms of the series $v_j^n$ are determined from the variational equality (4.2) for known $\bar{\sigma}_j^n$. It can be shown (see the Appendix) that, when $\rho < 2\alpha/\Delta$, the series $\bar{\sigma}_j^n$ converges to $\bar{\sigma}^n$, as $j \to \infty$.

Thus, Udzawa's algorithm reduces the solution of non-linear Problem 2 for unknown $\bar{\sigma}^n$ and $v^n$ to the successive determination of the quantities $\bar{\sigma}_j^n$ and $v_j^n$.

We will now determine $\bar{\sigma}_j^n$ from variational inequality (4.2). This problem, in turn, can be reduced to the problem of minimizing the quadratic function on the closed convex set $P_h$. Moreover, since each tensor function and the set $P_h$ are constant on each triangle $T_\nu^h$, it follows that finding $\bar{\sigma}_j^n$ reduces to the successive solution of problems of minimization on the closed convex set $B \subset R^6 \times R$ for each triangle $T_\nu^h$.

It is also possible to use another method. It is based on the fact that the variational inequality (4.2) is equivalent to the non-linear equation

$$x = \pi_P\{x - \bar{\rho}[\bar{A}(x - \bar{\sigma}^{n-1})/\Delta - \bar{e}(v_{j-1}^n)]\}, \quad \bar{A} = \begin{Vmatrix} A & 0 \\ 0 & \gamma \end{Vmatrix}, \quad \bar{e} = \begin{Vmatrix} e(v) \\ 0 \end{Vmatrix} \tag{4.3}$$

where $x = \bar{\sigma}_j^n$, $\bar{\rho}$ is an iterational parameter and $\pi_P = \pi_P(\bar{\sigma})$ is the operator of projection onto the closed convex set $P_h$.

We recall that $\pi_P(\bar{\sigma})$ is the projection of the 'vector' $\bar{\sigma}$ onto the set $P_h$ if the condition

$$\|\bar{\sigma} - \pi_P(\bar{\sigma})\| = \min_{\bar{\tau} \in P_h} \|\bar{\sigma} - \bar{\tau}\|$$

is satisfied. If the set $P_h$ is closed and convex, then such an element $\pi_P(\bar{\sigma})$ exists and is unique.

The solution of non-linear Eq. (4.3) is constructed by the method of successive approximations. The series $\{\bar{\sigma}_l\}$ $(l=1, 2, \ldots)$ is determined, where $\bar{\sigma}_l$ is calculated by means of the formula

$$\bar{\sigma}_l = \pi_P\{\bar{\sigma}_{l-1} - \bar{\rho}[A(\bar{\sigma}_{l-1} - \bar{\sigma}^{n-1})/\Delta - e(v_{j-1}^n)]\} \tag{4.4}$$

Since the stresses and the hardening parameter are piecewise constant, the projection operator acts on the set $B \subset R^6 \times R$ for each triangle $T_\nu^h$ of division. The series $\{\bar{\sigma}_l\}$ converges as $l \to \infty$ and $\bar{\sigma}_l \to \bar{\sigma}_j^n$. If the accuracy is specified using a certain number $\delta$, $0 < \delta < 1$, then the iteration process will end at a certain $\bar{l}$, and it is then possible to take $\bar{\sigma}_j^n \approx \bar{\sigma}_{\bar{l}}$.

The quantity $v_j^n$ is determined from relation (4.2). This problem is equivalent to solving a linear system of algebraic equations, and here the matrix of the coefficients of this system is the same whatever the values of $n$ and $j$. It is then reasonable to calculate the inverse matrix and to determine all $v_j^n$ by multiplying the same matrix by a vector, which depends on the numbers $j$ and $n$.

## 5. Example

As an example, we will consider the one-dimensional problem of the quasi-static deformation of an isotropic cylindrical tube under the action of a uniformly distributed load $P_0(t)$ applied to the inner surface of the tube. We will consider the motion of the tube in a cylindrical system of coordinates $r$, $\varphi$. In this case, only one component of the velocity vector, $v = v_r$, two components of the strain rate tensor, $e_r = \partial v/\partial r$ and $e_\varphi = v/r$, and three components of the stress tensor, $\sigma_r$, $\sigma_\varphi$ and $\sigma_z$, are non-zero. The tensor $A\sigma$ also has three non-zero components

$$(A\sigma)_\xi = (1+\nu)\left[\frac{\sigma_\xi}{E} - \frac{\sigma_r + \sigma_\varphi + \sigma_z}{3}\right], \quad \xi = r, \varphi, z \tag{5.1}$$

The bilinear form $A = A(\sigma, \tau)$ is then written as follows:

$$A(\sigma, \tau) = \frac{1+\nu}{E}\int_{r_1}^{r_2} r\left(\sigma_r\tau_r + \sigma_\varphi\tau_\varphi + \sigma_z\tau_z - \frac{\nu}{1+\nu}(\sigma_r + \sigma_\varphi + \sigma_z)(\tau_r + \tau_\varphi + \tau_z)\right)dr$$

The variational equality (4.2) can be reduced to the form

$$\int_{r_1}^{r_2}\left(\frac{\partial v_j}{\partial r}\frac{\partial w}{\partial r} + \frac{v_j w}{r^2}\right)rdr = P_0(t) - \int_{r_1}^{r_2}\left(\sigma_r^n\frac{\partial w}{\partial r} + \sigma_\varphi^n\frac{w}{r}\right)rdr, \quad v_j = v_j^n - v_{j-1}^n \tag{5.2}$$

Changing to an iteration procedure to solve the finite-dimensional problem, we will introduce in the segment $[1, r]$ the network $\Delta_h$, i.e. the division of the segment into $n$ parts with nodes $1 = r_0 < r_1 < \ldots < r_n = r$. On the network $\Delta_h$, the space of piecewise-linear functions $V_h$ and the space $H_h$ of piecewise-constant tensor fields $\sigma_h = (\sigma_r, \sigma_\varphi, \sigma_z)_h \in H_h$ are specified. It is obvious that any element of $V_h$ is clearly defined using $(n+1)$ values of the velocity at the nodes of the network, while any element of $H_h$ is clearly defined using $3n$ values of the stress tensor on each segment of the division.

From variational equality (5.2) there follows a system of equations that enable us to determine the unknown velocity values at nodes of the network. It is possible to check that the matrix of this system is a symmetric matrix and possesses the property of predominance of diagonal elements. Therefore, to solve the system, it is best to use the elimination method, which, as is well known, is stable and effective. If account is taken of the fact that the matrix of this system is independent of the iteration number or of the parameters $\rho$ and $\bar{\rho}$, it can be assumed that the proposed algorithm will also be stable and effective.

The stress tensor at each iteration is determined from relation (4.4). Conventionally, the calculations can be divided into two stages:

(1) calculation of the quantity in the braces in formula (4.4);
(2) projection of this quantity onto the set $P_h = H_h \cap P$.

As pointed out above, the second stage reduces to solving a problem of non-linear programming on each element of the network in the space of the variables $(\sigma_r, \sigma_\varphi, \sigma_z, \chi)$. To solve this problem, effective numerical methods have been established.[18,19] Note that the second stage is the most laborious part of the algorithm for solving the quasi-static problem. The proposed method is one of the possible methods for solving variational inequalities. Other methods can also be used, but they all somehow or other reduce the solution of the variational inequality to a certain iteration procedure.

A method involving the operation of projection was chosen here only in order to enable us to compare this method with other methods where the procedure of stress correction is used. The method used to solve variational inequality (4.1) has a rigorous basis, but the number of iterations within which the solution of the problem of non-linear programming can be constructed depends on the spcified accuracy. In this respect, the method differs from other methods based on the procedure of stress correction, where the procedure itself consists of a certain prescribed number of specified actions.

## 6. Appendix

We will show that the iteration process constructed using Udzawa's algorithm reduces to solving a finite-dimensional problem. To this end, in the first inequality of (3.1) we will take $\tau = \bar{\sigma}_j^n$ as a trial function, and in inequality (4.1) $\tau = \bar{\sigma}^n$. We then add the two inequalities and obtain

$$[(\bar{\sigma}_j^n - \bar{\sigma}^n), (\bar{\sigma}^n - \bar{\sigma}_j^n)] - \Delta(e(\upsilon_{j-1}^n - \upsilon^n):(\sigma^n - \sigma_j^n)) \geq 0 \tag{6.1}$$

The first term on the left-hand side of inequality (6.1) is a scalar product associated with the symmetric bilinear norm

$$[\bar{\sigma}, \bar{\tau}] = (A\sigma, \tau) + \chi\gamma\eta, \quad \bar{\sigma} = (\sigma, \chi), \quad \bar{\tau} = (\tau, \eta)$$

The norm related to it will be denoted by $\langle\bar{\sigma}\rangle \equiv [\bar{\sigma}, \bar{\sigma}]^{1/2}$. We then multiply the second equation of (3.1) by $\rho$ and subtract relation (4.2) from it. We introduce $w_j = \upsilon_j^n - \upsilon^n$ as the trial function $w$ and obtain

$$(e(w_j):e(w_j)) = (e(w_{j-1}):e(w_j)) + \rho((\sigma^n - \sigma_j^n):e(w_j)) =$$
$$= ((e(w_{j-1}) + \rho(\sigma^n - \sigma_j^n)):e(w_j)) \leq \left\|e(w_{j-1}) + \rho(\sigma^n - \sigma_j^n)\right\| \left\|e(w_j)\right\|$$

The latter estimate follows from the Cauchy inequality. Cancelling the common factor on both sides of the inequality, we arrive at the relations

$$\begin{aligned} \|e(w_j)\|^2 &\leq \left\|e(w_{j-1}) + \rho(\sigma^n - \sigma_j^n)\right\|^2 = \\ &= \|e(w_{j-1})\|^2 + \rho^2\left\|\sigma^n - \sigma_j^n\right\|^2 + 2\rho(e(w_j):(\sigma^n - \sigma_j^n)) \end{aligned} \tag{6.2}$$

The last term on the right-hand side can be eliminated if inequality (6.1) is used. Multiplying the inequality (6.1) by $2\rho/\Delta$ and adding with equality (6.2), we obtain

$$\|e(w_j)\|^2 + 2(\rho/\Delta)\langle\sigma^n - \sigma_j^n\rangle^2 \leq \|e(w_{j-1})\|^2 + \rho^2\left\|\sigma^n - \sigma_j^n\right\|^2 \leq$$
$$\leq \|e(w_{j-1})\|^2 + (\rho^2/\alpha)\langle\sigma^n - \sigma_j^n\rangle^2$$

since $||\sigma^n - \sigma_j^n||^2 \leq \langle\bar{\sigma}^n - \bar{\sigma}_j^n\rangle^2/\alpha$. We now combine like terms and sum over $j$ from unity to a certain $M$. Changing from the functions $w_{j-1}$, $w_j$ to the functions $\upsilon_{j-1}^n$, $\upsilon_j^n$, we obtain

$$\left(\frac{2\rho}{\Delta} - \frac{\rho^2}{\alpha}\right)\sum(M) + \|e(w_M)\|^2 \leq \|e(w_0)\|^2, \quad \sum(M) = \sum_{j=1}^{M}\langle\bar{\sigma}^n - \bar{\sigma}_j^n\rangle^2$$

We select $\rho$ such that the factor before the summation sign is positive, i.e. $\rho < 2\alpha/\Delta$. Then, from the last inequality it follows that the series $\sum(\infty)$ converges, and so $\langle\bar{\sigma}^n - \bar{\sigma}_j^n\rangle \to 0$ when $j \to \infty$. This proves the convergence of the iteration process to the solution of the finite-dimensional problem.

[Aleksandr Nikolayevich Kovshov (1941–2002), senior research fellow of the Institute of Problems in Mechanics of the Russian Academy of Sciences, Candidate of Phys.-Math. Sci., and author of over fifty papers on the mechanics of deformable rigid bodies, took part in the discussion of the concept and initial versions of this paper. We dedicate it to his memory.]

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